A heuristic approach to the matrix apportionment problem

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1. Description of the problem

The matrix apportionment problem deals with the assignment of seats to individual party lists in multimember districts or constituencies. The solution sought after is the optimal solution of a certain entropy optimization problem, i.e. this one:

(1.1) $$max \sum_i \sum_j \sum_k \ln\left({V_{ij}}/{d_k} \right) x_{ijk}$$

subject to

(1.2) $$\sum_j \sum_k x_{ijk} = C_i \ \forall\ i$$

(1.3) $$\sum_i \sum_k x_{ijk} = P_j \ \forall\ j$$

(1.4) $$x_{ijk} = 1 \text{ for } k = 1,2,\dots,m_{ij} \ \forall\ i,j$$

(1.5) $$x_{ijk} = 0 \text{ or } 1 \ \forall\ i,j,k$$

Here

- i index for the constituencies,
- j index for the parties,
- L_{ij} a list (or slate),
- k a running index, 1, 2, 3, etc. for the number of the seats of a list,
- V_{ij} is the number of votes of the list L_{ij},
- d_k an increasing series of positive divisors; i.e. $d_k = k$ for the d'Hondt's rule,
- C_i the total number of seats of constituency i,
- P_j the total number of seats of party j,
- x_{ijk} is 1 if list L_{ij} is assigned its k'th seat but 0 zero otherwise,
- m_{ij} an eventually preassigned number of seats to list L_{ij}

and

(1.6) $$A := \sum_i \sum_j \sum_k x_{ijk} = \sum_i C_i = \sum_j P_j$$

is the total number of seats in the parliament.

Obviously in the optimal solution satisfies

(1.7) $$x_{ijk} \geq x_{ijk+1}$$

so there are no „missing“ seats in between.

The final number of seats assigned to list L_{ij} will be equal to

(1.8) $$\sum_k x_{ijk}$$

Constraints (1.5) can be relaxed to

(1.9) $$0 \leq x_{ijk} \leq 1 \; \forall \, i,j,k$$

as the set of constraints is unimodular.

For later use we will call the value V_{ij}/d_k the votes per the k'th seat of the list L_{ij} or just the *seat-votes*.

2. Heuristic approach

There is no simple greedy algorithm for solving the original problem (1.1-5); the complexity is larger than that. However, for politicians in general only greedy algorithms are acceptable.

The above approaches do not lead to any simpler problems than the original one, (1.1-5). Relaxing the total number of seats leaves us with a primal problem of (almost) the same complexity as the original one whereas in relaxing the party or constituency constraints the primal is easy but the dual difficult.

If looking for a greedy solution one has to be satisfied with an approximate solution. Thus we are looking for a heuristic, greedy approach. We will focus here on a promising one, which we call the *Relative Superiority Algorithm*, here abbreviated as RSA. The basic idea behind RSA is the Vogel-heuristic for the transportation problem. Directly transferred to the matrix apportionment problem the Vogel-analogy would prior to each assignment measure the superiority, in some sense, of the strongest candidate over other candidates both in the same constituency and the same party. Then the next seat would be assigned to the strongest candidate with the highest superiority.

However, for not making the apportionment algorithm too complicated we focus only on one of these dimensions and choose the constituency superiority. It is our feeling that for the politicians a satisfying or „normal" assignment of the seats in the constituency is more important than how the seats are distributed within the parties.

Looking for a sensible measure of superiority we argue as follows. If the currently strongest available candidate in a constituency will not be given a seat then later in the apportionment process a candidate, that now would be a candidate for a substitute seat, would have to be assigned a permanent seat to fulfill the requirement of the constituency for full number of seats. Thus we measure the superiority of the currently strongest candidate over the (first) substitute candidate as the difference of the logarithm of the seat-votes of these two candidates, the strongest one and the substitute. Instead of calculating the difference of the logarithms one can just as well calculate the ratio of the seat-votes of these two.

The following is more precise description of the algorithm:

Relative Superiority Algorithm (RSA)

1. At each stage of the apportionment process the following must be calculated for all those constituencies that have not had all its seats filled:
    a. Calculate the seat-votes of all next candidates of those lists that are still eligible for further seats.
    b. Within each constituency sort the candidates in descending order by the number of their seat-votes until the first candidate for a substitute seat has been found, but taking care of the following:
        i. All further candidates of the list of the highest ranking candidate of the constituency are ignored.
        ii. As soon as in this provisional apportionment the seat-requirement of a party (including the already permanently assigned seats) has been fulfilled any further candidates of that party are ignored. (But of course this applies only to one constituency at a time.)
    c. Calculate the ratio of the seat-votes of the top candidate to that of the first candidate for a substitute seat in the constituency. This ratio will be called the *relative superiority* of the top candidate.
2. Apportion the next seat permanently to that top candidate in a constituency which has the highest relative superiority.
3. Repeat these steps until all seats have been apportioned.

The following figure shows how the relative superiority is calculation in one of the constituencies in the Icelandic election 2013.

In assigning adjustments seats (on the top of the fixed constituency seats) in the four Icelandic elections in this century the RSA finds the optimal solution in three cases. Only in 2007 it misses the best solution, but comes very close to it.

Calculation of the relative superiority in the South-West Constituency in Iceland 2013.
Symbols refer to the parties.
There are still two seats to be apportioned.