Bidimensional Election Systems:

Apportionment methods in theory and practice

TALK AT TECHNICAL UNIVERSITY OF MUNICH ON JANUARY 14, 2019

(FORSCHUNGSSEMINAR M9: ANGEWANDTE GEOMETRIE UND DISKRETE MATHEMATIK)

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Some of the following slides were not shown in the talk, due to lack of time

Abstract

In the Nordic countries, as well as in many other, the electoral systems to the national assemblies are in a way bidimensional: Seats are apportioned within constituencies (or districts) but with respect to the national outcome of the parties. For that purpose, the seats are divided into proper constituency seats and adjustment seats. The allocation of the latter is mathematically interesting but politically controversial.

Balinski and Demange have presented fairness properties which allocation methods of this kind (i.e. of the adjustment seats) should respect. They prove that this demand leads to one and only one method – given a specific underlying one-dimensional divisor rule like that of d'Hondt or Sainte-Laguë. This optimal solution can also be formulated as a simple linear optimal assignment problem. Pukelsheim has managed to convince law makers in the Canton of Zürich, to adopt this optimal allocation method (based on dual multipliers).

The speaker, who has been advising the Parliament and Governments in Iceland (one of the Nordic countries) for over a quarter of a century on electoral systems, has however experienced that politicians, lawyers and political scientists will only accept recursive algorithms for seat apportionments. Iterative methods are not agreeable but the optimal solution calls for iterations. Consequently, practical allocation methods for bidimensional electoral systems are inevitably only approximations to the optimal method.

In the talk several near optimal allocation methods will be presented, many of which are derived from heuristics for the classical transportation problem (Monge, Vogel). To test the practicality and quality of these methods a simulation model has been developed and is presented in the talk. This model generates random election outcomes (with user-given averages, e.g. actual or typical election results). The seats are then allocated using the different heuristic methods. The quality of each method is measured using different indicators, classical and new, thus enabling a ranking of the tested methods, in particular in comparison with the optimal method.

Iceland, constituencies

The bidimensional proportional apportionment problem

as in Iceland, Norway, Sweden, Denmark (3D) and many other (mainly European) countries, even the election to the *Bundestag*...

Constituencies

(generally multimember)

- Parties put up lists or candidate(s)
- Proper constituency seats
- Apportioned independently on the basis of the outcome in each constituency

Adjustment seats

(also compensatory or equilization seats)

- Distributed by law prior to the election among the constituencies
- Apportioned to the parties proportionally based on the national outcome
- **Then allocated „proportionally" to the individual party lists**

Iceland 2003	Parties						
Constituencies	**B**	**D**	**F**	**S**	**U**	**Seats in total**	
Norðvestur						**10**	C_1
Norðaustur						**10**	
Suður			V_{ij} x_{ij}			**10**	**...**
Suðvestur						**11**	
Reykjavík suður						**11**	
Reykjavík norður						**11**	C_m
Seats by parties (const. and adjustment)	**12**	**22**	**4**	**20**	**5**	**63**	
	P_1		**...**		P_n		A

The problem:

Allocate seats x_{ij} s.t. vertical and horizontal sums are as given and that the assignments are as far as possible proportional to the numbers of votes V_{ij} to the lists

Complications

- In general $\sum_i C_i = A = \sum_j P_j$ (not so in the *Bundestag* election)
  - Inequalities can be dealt with but I will in this talk stick to equalities
- Most often the (preassigned) constituency seats prescribe lower limits, m_{ij} requiering $x_{ij} \geq m_{ij}$
  - This may lead to „overhang“ seats, which is politically problem, although technically not so
- Generally the seats P_j are allocated proportionally to the national outcome
- This may not be the case for the (prior) distribution of seats C_i to the constituencies
  - In Iceland the disproportionality is up to 2:1
  - In Norway the distribution is based on

$$Number\ of\ inhabitants + 1.8 * Squarekilometers$$

  - In Denmark:

$$Number\ of\ inhabitants + Number\ of\ reg.voters + 20 * Squarekilometers$$

Allocation of adjustment seats

Balinski and Demange have proved:

There is only one solution to the bidimensional problem (given an underlying divisor rule, like d'Hondt's rule) satisfying some sensible axioms, like these (here somewhat freely interpreted):

- *Monotonicity*: No list looses seat by getting more votes or vice versa
- *IIA*: Changes in votes of lists not leading to changes in allocation to them shall not affect allocation to other lists

Michel Louis Balinski / Gabrielle Demange: «An axiomatic approach to proportionality between matrices.» Mathematics of Operations Research 14 (1989) 700-719

An objective reflecting some kind of proportionality of the matrix of allocated seat (x_{ij}) to the matrix of votes (V_{ij})

subject to the constraints

(1) $\sum_j x_{ij} = C_i$

(2) $\sum_i x_{ij} = P_j$

(3) $x_{ij} \geq m_{ij}$

(4) x_{ij} integer

i	index for the constituencies
j	index for the parties
C_i	total number of seats of constituency i (proper constituency seats as well as adjustment seats)
P_j	total number of seats of party j
V_{ij}	votes of list L_{ij}
m_{ij}	number of preassigned seats to list L_{ij} e.g. number of constituency seats already assigned
x_{ij}	number of seats to be allocated to list L_{ij}

Optimal solution

A result of Balinski and Demange can be interpreted so:

The only bidimensional apportionment method (given the divisor rule) satisfying (1-4) and fullfilling the axioms of B&D is equvalent to the shown linear optimization problem

(The solution will be integer!)

Coworker: Prof. Kurt Jörnsten

Thorkell Helgason / Kurt Jörnsten: «Entropy of proportional matrix apportionments.» Norwegian School of Economics and Business Administration, Institute of Finance and Management Science, Working Paper 4/94. Bergen-Sandviken, 1994.

(5) $$\max \left[\sum_i \sum_j \sum_k \ln \left({V_{ij}}/{d_k} \right) x_{ijk} \right]$$

subject to the constraints

(6) $$\sum_j \sum_k x_{ijk} = C_i$$

(7) $$\sum_i \sum_k x_{ijk} = P_j$$

(8) $$0 \le x_{ijk} \le 1$$

(9) $$x_{ijk} = 1 \text{ for } k = 1, \dots, m_{ij}$$

Votes *V* can be replaced with any other measure of the outcome of the election, e.g. the number of seats the lists would get in a pure constituency allocation

d_k an increasing series of positive divisors; i.e. $d_k = k$ for the d'Hondt's rule

x_{ijk} is 1 if list L_{ij} gets its k'th seat assigned but 0 zero otherwise

Alternating scaling method

By relaxation of the main constraints (6) and (7) one easily finds out that the optimal solution amounts to find optimal (dual) multipliers

Biproportional matrix scaling and the iterative proportional fitting procedure

F Pukelsheim - 2013 - opus.bibliothek.uni-augsburg.de

Find optimal (dual) multipliers α_i and β_j and rescale the votes

$$W_{ij} := {}^{V_{ij}}\!/_{\alpha_i \beta_j} \tag{10}$$

after which the apportionment based on the chosen divisor rule along parties or constituencies yield the same matrix apportionment satisfying the common constraints (6-9)

One set of the multipliers (alfas or betas) suffices. Needing only one dimension may be politically easier

Constituency relaxation

Relaxation of the constituency constraints (6) alone leaves us with only one set of (unknown) multipliers

Find optimal (dual) constituency multipliers α_i and rescale the votes

$$W_{ij} := {}^{V_{ij}}/_{\alpha_i} \qquad (11)$$

after which the apportionment based on the chosen divisor rule along parties, given the party constraints (7), also satisfies the constraint on the constituency sums (6)

Happy end?

For mathematicians the relevant paragraph of an election act could be quite simple:

Apportionment of [adjustment] seats to individual lists:

1. *Determine allocation quotients by dividing the votes of the list by the integers 1, 2, 3. etc. [d´Hondt].*
2. *Apportion seats such that the product of the corresponding allocation quotients is maximized provided*
   a) *that the total number of seats in each constituency equals the number of seats prescribed [see a previous paragraph] and*
   b) *the total number of seats for each party equals the number of seats already attributed to it [see a previous paragraph].*
3. *[If ties then ...]*

Has the optimal allocation method been implemented or if not why not?

Yes

- Cantons in Switzerland, first in Zürich
  - ✓ Thanks to Pukelsheim

No

- Not so in the Nordic countries
  - ✓ Although it has been suggested and advocated

Political restrictions of mandate apportionments

At least in the Nordic countries

Under-standable

- To politicians, lawyers, political scientists; even to the man on the street
  - See e.g. *Bundesverfassungsgericht* 2008

Recursive

- Seats to be assigned one by one
  - No regrets, i.e. no iterations

One-dimensional divisor rules:

Two ways of presentation

For later use:
These are the "ideal" or "fair" shares of seats

E.g. in the Nordic Countries:

d'Hondt, Sainte-Laguë ...

- Primal: Constructive, one-by-one (here d'Hondt)

Parties	A	B	B	D	Sum
Votes / Divisors	**4.500**	**3.400**	**1.400**	**600**	**10.000**
1	**4.500**	**3.400**	**1.400**	600	
2	**2.250**	**1.700**	700		
3	**1.500**	**1.133**			
4	**1.125**	**850**			
5	**900**	680			
6	750				
Apportioned	**5**	**4**	**1**		**10**

Elsewhere

Jefferson, Webster, Bischoff, Schepers ...

- Dual: Guessing a quota (here d'Hondt)

Parties	A	B	B	D	Sum
Votes	**4.500**	**3.400**	**1.400**	**600**	**10.000**
First guess of quota (Hare)					1000
Seat shares	4,50	3,40	1,40	0,60	
Round down	4	3	1		8
Second guess of quota (Droop)					909
Seat shares	4,95	3,74	1,54	0,67	
Round down	4	3	1		8
Final guess of quota					800
Seat shares	5,63	4,25	1,75	0,88	
Round down	**5**	**4**	**1**		**10**

One-dimensional divisor rules:

Two ways of presentation

For later use:
These are the (one-dimensional) “ideal” shares of seats

E.g. in the Nordic Countries:

d‘Hondt, Sainte-Laguë ...

- Primal: Constructive, one-by-one (here d‘Hondt)

Parties	A	B	B	D	Sum
Votes / Divisors	**4.700**	**3.400**	**1.400**	**500**	**10.000**
1	**4.700**	**3.400**	**1.400**	500	
2	**2.350**	**1.700**	700		
3	**1.567**	**1.133**			
4	**1.175**	**850**			
5	**940**	680			
6	783				
Apportioned	**5**	**4**	**1**	**0**	**10**

Elsewhere

Jefferson, Webster, Bischoff, Schepers ...

- Dual: Guessing a quota (here d‘Hondt)

Parties	A	B	B	D	Sum
Votes	**4.700**	**3.400**	**1.400**	**500**	**10.000**
First guess of quota (Hare)					1000
Seat shares	4,70	3,40	1,40	0,50	
Round down	4	3	1	0	8
Second guess of quota (Droop)					909
Seat shares	5,17	3,74	1,54	0,55	
Round down	5	3	1	0	9
Final guess of quota (in the range 784-850)					800
Seat shares	5,88	4,25	1,75	0,63	
Round down	**5**	**4**	**1**	**0**	**10**

Morale

- In the Nordics the electorate and candidates are used to seeing step by step what happens
- In the German-speaking world the public is used to be presented with the results which they can verify (by playing with the quota)
  - Therefore, may be, Pukelsheim has succeeded with the alternating scaling in the Switzerland
- But neither I, nor my colleagues in other Nordic countries
- In Italy, Serafini et.al. have suggested a compromise:
  - A „Solver“ (Virgil) presents the solution, the apportionment
  - A „Verifier“ (Dante), as a layman, checks the validity
- The one-dimensional „Constituency relaxation“ might fit into this middle-road

So what now?

Find a good heuristic to the B&D optimal method

Better than the current election laws

But what is good?

Measures
of proportionality

Distance from some reference apportionments

Measured as number of different assignments

The minimum is 4 and then always even numbers

Reference apportionments:

- Optimal method (B&D=Max. entropy=AS)
- Current Icelandic election act
- All seats constituency seats, i.e. no adjustment seats

Some quality indices

Some of many proportionality indices suggested for the one-dimensional case; here adapted to two dimensions

In one dimension:

- Laguë minimizes *LaguëSum*
- d´Hondt minimizes *d´HondtSum*
- d´Hondt maximizes *d´HondtMin*

$$Loosemore\&Hanby := \sum_i \sum_j \left| S_{ij} - x_{ij} \right|$$

$$LaguëSum := \sum_i \sum_j \frac{\left(S_{ij} - x_{ij}\right)^2}{S_{ij}}$$

$$d'HondtSum := \sum_i \sum_j \frac{\left(S_{ij} - x_{ij}\right)^+}{S_{ij}}$$

$$d'HondtMin := \min_{i,j} \frac{S_{ij}}{x_{ij}}$$

$$Ideal\ share\ of\ seats := S_{ij} := {}^{V_{ij}}/_{\rho_i \mu_j}$$

s.t.

$$\sum_j S_{ij} = C_i$$

$$\sum_i S_{ij} = P_j$$

ρ_i and μ_j non-negative reals

Methods
for allocating (adjustment) seats

(Few out of several tested)

A handy definition

- In the following methods seats are assigned one by one
- After each assignment, say to list L_{ij}, we will update the preassigned number of seats to that list: $m_{ij} \rightarrow m_{ij} + 1$
- Therefore, at each step we focus on candidate no. $m_{ij} + 1$
- We will therefor refer to

Seat-value of the next candidate$:= N_{ij} := \dfrac{V_{ij}}{d_{m_{ij}+1}}$

or simply as the *Seat-value* of the list or the next candidate

Icelandic system

- **54 constituency seats**
  - 6 constituencies (now) with 7-11 seats each
  - Apportioned in each by d’Hondt
- **9 adjustment seats to parties over 5%**
  - One in 3 constituencies and two in the other 3
  - Apportioned nationally by d’Hondt, seat by seat:
  - At each step find that list of the relevant party whose next seat-value is highest as percentage of valid votes in the corresponding constituency
  - Assign the next seat to this list

Icelandic system simplified

- **54 constituency seats**
  - 6 constituencies with 7-11 seats each
  - Apportioned by d'Hondt
- **9 adjustment seats to parties over 5%**
  - One in 3 constituencies and two in the other 3
  - Apportioned nationally by d'Hondt
  - Find the list with the highest seat-value as percentage of valid votes in the corresponding constituency
  - Assign the next seat to this list

Norwegian system

- **150 constituency seats**
  - 19 constituencies
  - Apportioned by the Scandinavian Sainte-Laguë (first divisor 1.4, not 1, but then 3, 5 …)
- **19 adjustment seats to parties over 4%**
  - One per constituency
  - Apportioned nationally to parties by Sainte-Laguë
  - Find the list with the highest ideal constituency-share (i.e. calculated one-dimensionally)
  - Assign the next seat to this list

A basic idea:

2 + 2 alternating chains

A tentative solution to the biproportional optimization can be improved if an augmenting alternating chain can be found

Looking for such chains is – politically – impossible

But the shortest chains involving only 2+2 lists may be acceptable

- Consider four (available lists), located in a rectangle in the table of lists:

L_{ij}	…	L_{il}
…		…
L_{kj}	…	L_{kl}

- If $N_{ij}\ N_{kl} > N_{kj}\ N_{il}$ (recall $N := \frac{V}{d_{m+1}}$)

  then allocating seats to the pair L_{ij} and L_{kl} would contribute more to the entropy than if we would choose the other diagonal pair L_{kj} and L_{il}

Monge

A.J. Hoffman, referring to an 18th century scholar Gaspar Monge, identified conditions under which a route (a cell) in the Transportation problem must always be included in the optimal solution

This can be translated into the Entropy Optimization apportionment; not shown here

But nevertheless here is a simplified idea

- Consider a list L_{ij} as a candidate for the next seat in a recursive procedure
- Consider four (available lists), located in a rectangle in the table of lists:

L_{ij}	…	L_{il}
…		…
L_{kj}	…	L_{kl}

- Define

$$\textit{Monge-value} := M_{ij} := \min_{k,l} \left\{ \frac{N_{ij}\, N_{kl}}{N_{kj}\, N_{il}} \right\}$$

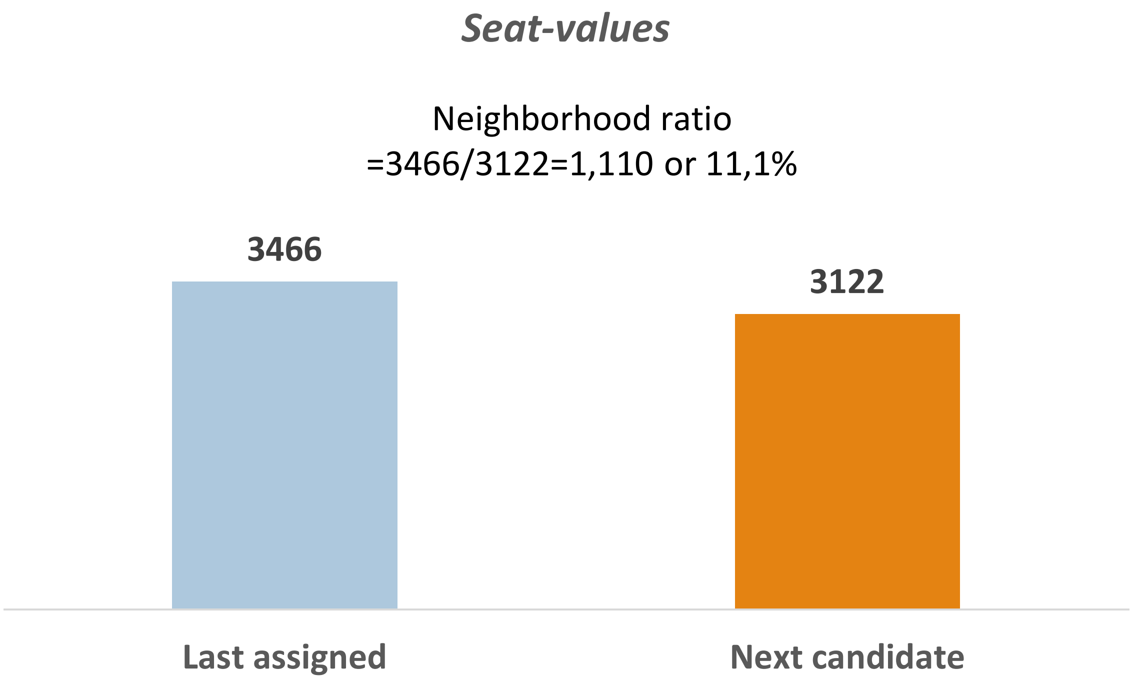
Recall the seat-values: $N := \frac{V}{d_{m+1}}$

- ***Assign a seat to the list with highest Monge-value***

Nearest neighbor method

This is a simplified version of the Relative superiority algorithm

- In each constituency the next candidate is compared to the last assigned one
- This is done by comparing the ratios of their seat values
- Where this comparative ratio is highest the next seat is allocated



Relative superiority algorithm

<u>Motivation:</u>

Vogel's approximation in the Transportation problem

Why not, like by Vogel, also find the relative superiority within a party?

Too complicated:

- Calls for first scaling the votes
- Politicians cannot only digest one-dimension at a time!

- In each constituency the next top candidate is compared to the first possible substitute candidate to a seat

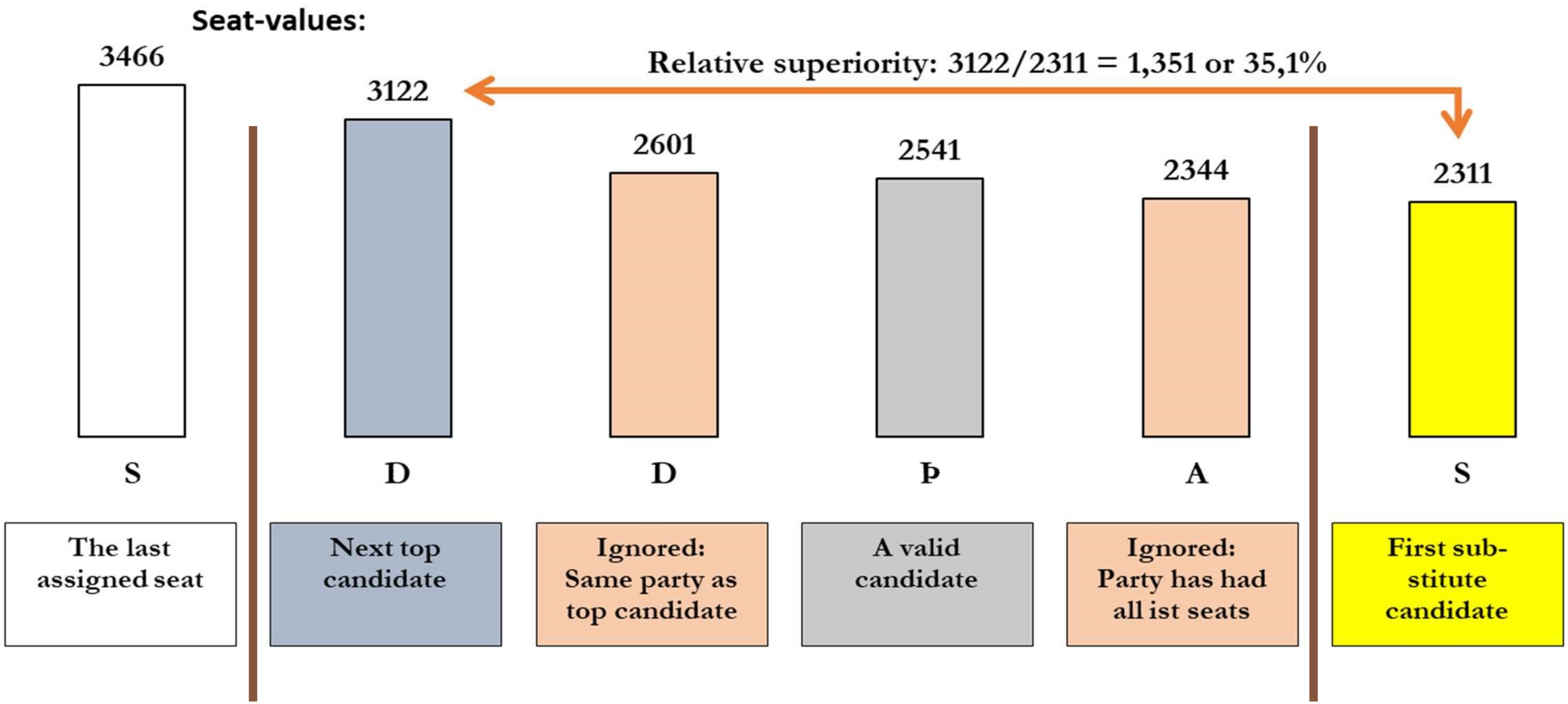
  Other candidates of the party to which the first candidate belongs are ignored

- This is done by comparing the ratios of their seat values
- Where this comparative ratio is highest the next seat is allocated

RSA Relative superiority algorithm

South-West Constituency in Iceland 2013

There are still two (adjustment) seats to be apportioned



Switching method

This method was suggested by the author in a revision of the Icelandic system in 1982 and was approved by the chairmen of all the major parties

However, it was killed in Parliament by a nickname, “The execution method”:

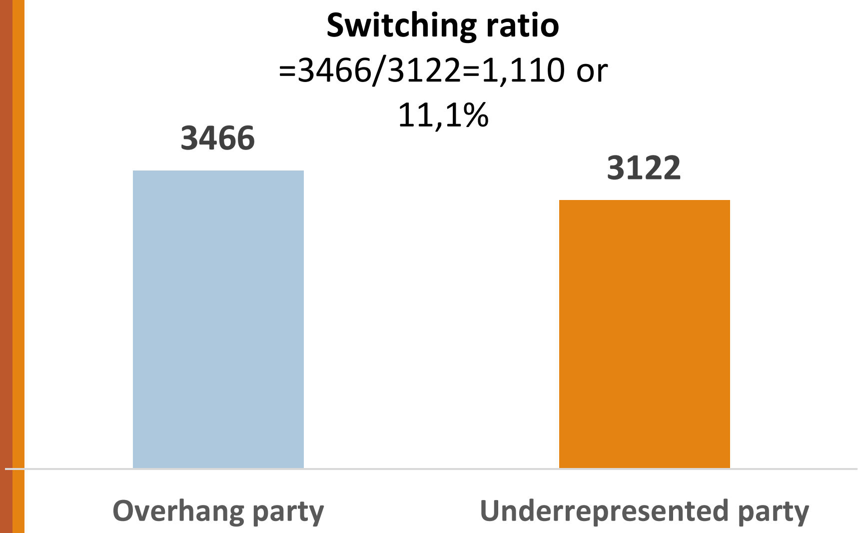
> “First you are elected in a constituency but then you are executed by an electoral squad”

A method, akin to this Switching method, where “overhang” seats in the allocation of the proper constituency seats are withdrawn, has been proposed 2008 in an official Swedish report; also in a paper by Ramírez-Gonzáles et. al. 2014

A similar method has (recently) also been advocated in Iceland by Mr. Kristinn Lund

- First all seats are apportioned as constituency seats
- Find how many seats nationally to the parties
  - Some may get an “overhang”, i.e. are “overrepresented”
  - Other are “underrepresentation”
- In each constituency:
  - Calculate: Current seat value of an overhang party / Next seat value of an underrepresented party
- Where this ratio is smallest, seats are switched

Seat-values in a particular constituency



- The idea is to minimize the change in entropy in this inevitable correction
- An other motivation is to minimize the deviations from (total) constituency apportionments

Heuristic methods and the B&D axioms

- Every heuristic method violates at least one of the axioms B&D; e.g. the monotonicity or the IIA
- But is it often or only in exceptional cases?
  - Under the current Icelandic election law six elections have been carried out
  - In two cases (i.e. by two lists) the monotonicity requirement was violated; this is out of some 250 possible cases, or about 1%
  - The IIA was much more often violated

Testing
the apportionment ideas

Which election data?

- Using real election results for testing the quality of proposed methods (election acts) has at least two drawbacks:
  - There are too few elections results for any reasonable statistics!
  - And passed elections are a part of history which may disturb the judgment of the decision makers (politicians)
- However election results used in tests to compare methods should be "relevant" in some sense
  - The best method for a system with just one adjustment seat in each constituency (Norway, UK, …) may not be the best for the other extreme, i.e. multimember constituencies where all seats are adjustment seats
- **So we need simulated election results for the tests**

Simulating election results

A work in progress!

Coworkers:

- Martha G. Bjarnadóttir
- Pétur G. Ólafsson
- Smári McCarthy

In alphabetic order, Icelandic style!

Note:

We are NOT fitting distribution to a data; just generating possible election outcomes

1. *Reference outcome*: A specific election outcome, or a "typical" one
   - Like an average of all six elections in Iceland in this century
2. *Distribution used*: Beta distribution for each cell (list)
   - Mean votes for a list = The votes in the reference outcome
   - A parameter controlling the standard deviation is inputted
   - The lists are considered independent from each other
3. *Number of simulations:*
   - 1000 iterations seem to be enough; takes about 2 min per method in the example with about 40 lists and 9 adjustment seats.

Voting Instructions Single Election Simulate

Simulate elections

Settings

Simulation settings

Number of simulations

1000

How many simulations should be run?

Generating method

Beta distribution

Which method should be used to generate random votes?

Stability parameter

100

To influence the standard deviation of the distribution, please provide a number greater than 1 (does not need to be an integer, and values close to 1 are allowed, such as 1.0001). This number represents stability, in some sense. Higher values result in lower standard deviation, and vice versa.

Simulate elections

Add election ruleset

Name

Iceland 2017 current allocation method

Give this rule set a name.

Delete this ruleset

Divider for allocating constituency seats

D'Hondt's method

Which divider rule should be used to allocate constituency seats to lists within each constituency?

Divider for apportioning adjustment seats

D'Hondt's method

Which divider rule should be used to apportion adjustment seats among parties?

Divider for allocating adjustment seats

D'Hondt's method

Which divider rule should be used to allocate adjustment seats to individual lists?

Adjustment method

Icelandic law 24/2000 (Kosningar til Alþingis)

Which method should be used to allocate adjustment seats?

Adjustment threshold

5 %

What threshold are parties required to reach to qualify for adjustment seats?

Programming language: *Python*

The terminology here is a bit different from that in the other slides. Will be synchronized!

Name

Iceland 2017 entropy optimal

Give this rule set a name.

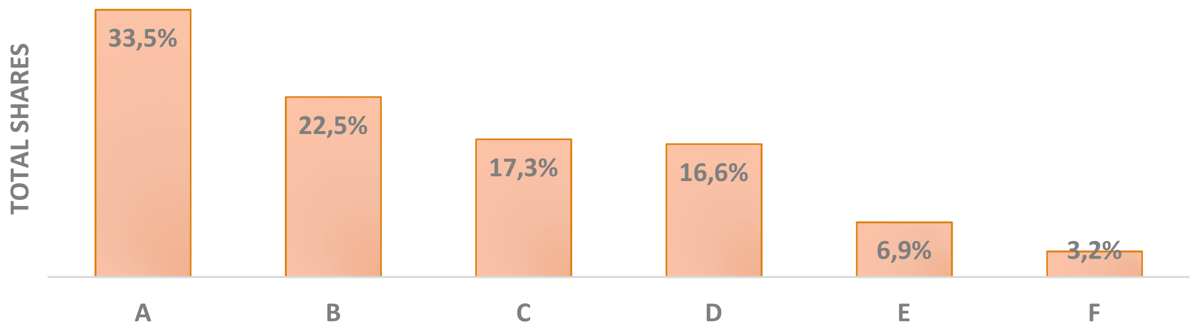
Delete this ruleset

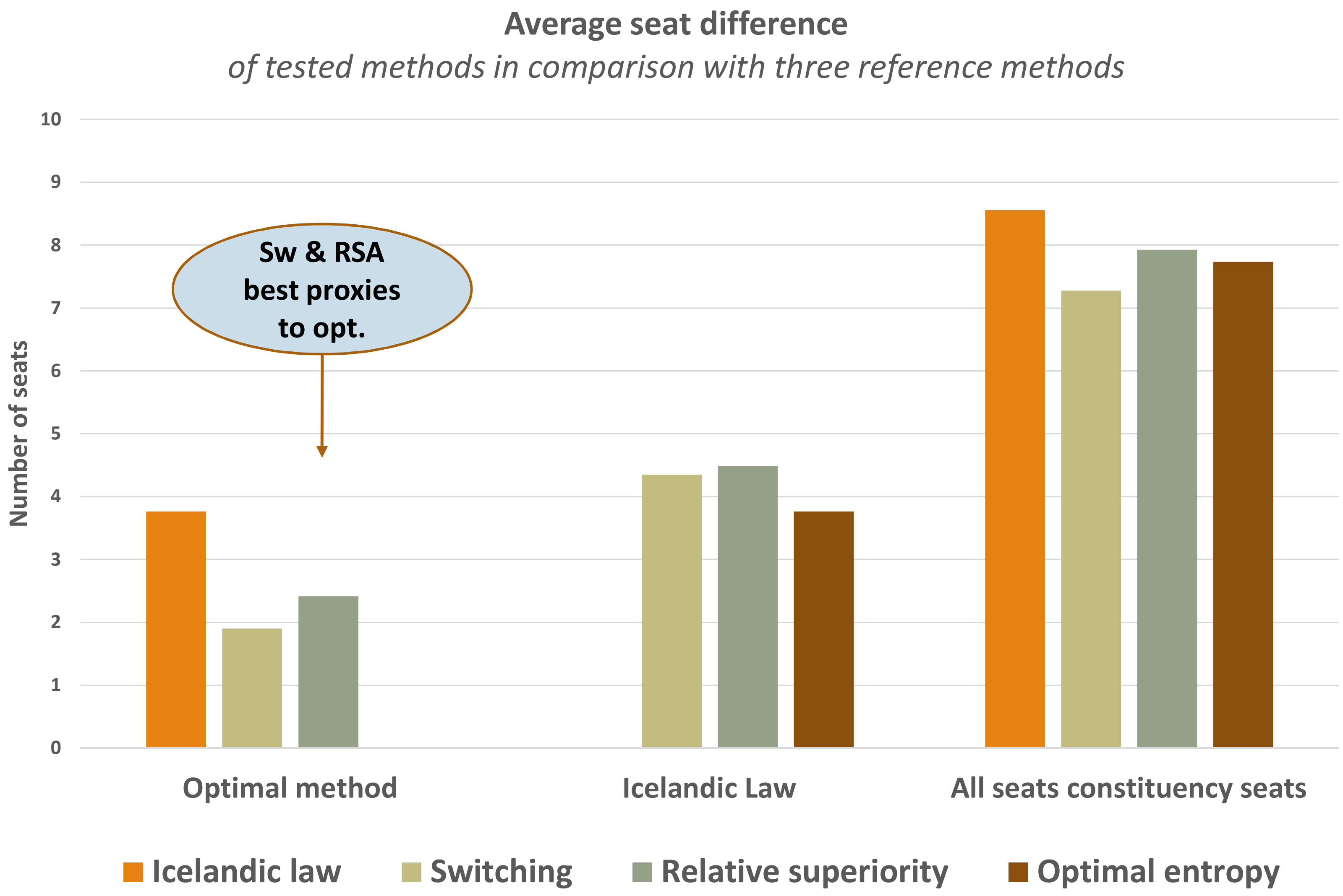
Some results
from the simulations

Reference data:

Sort of average of all six elections in Iceland in this century

	Const. seats	Adj. seats	A	B	C	D	E	F
Norðvestur	**7**	**1**	**30,8%**	**18,3%**	**26,6%**	**17,9%**	**4,6%**	**1,8%**
Norðaustur	**9**	**1**	**25,6%**	**18,3%**	**28,1%**	**22,0%**	**4,0%**	**2,0%**
Suður	**9**	**1**	**33,8%**	**21,2%**	**25,9%**	**11,3%**	**5,7%**	**2,2%**
Suðvestur	**11**	**2**	**38,9%**	**23,6%**	**13,5%**	**13,1%**	**6,9%**	**4,0%**
Reykjavík suður	**9**	**2**	**33,7%**	**24,6%**	**11,4%**	**18,2%**	**8,3%**	**3,7%**
Reykjavík norður	**9**	**2**	**31,1%**	**24,8%**	**10,5%**	**20,6%**	**9,3%**	**3,7%**
Total	**54**	**9**	**33,5%**	**22,5%**	**17,3%**	**16,6%**	**6,9%**	**3,2%**





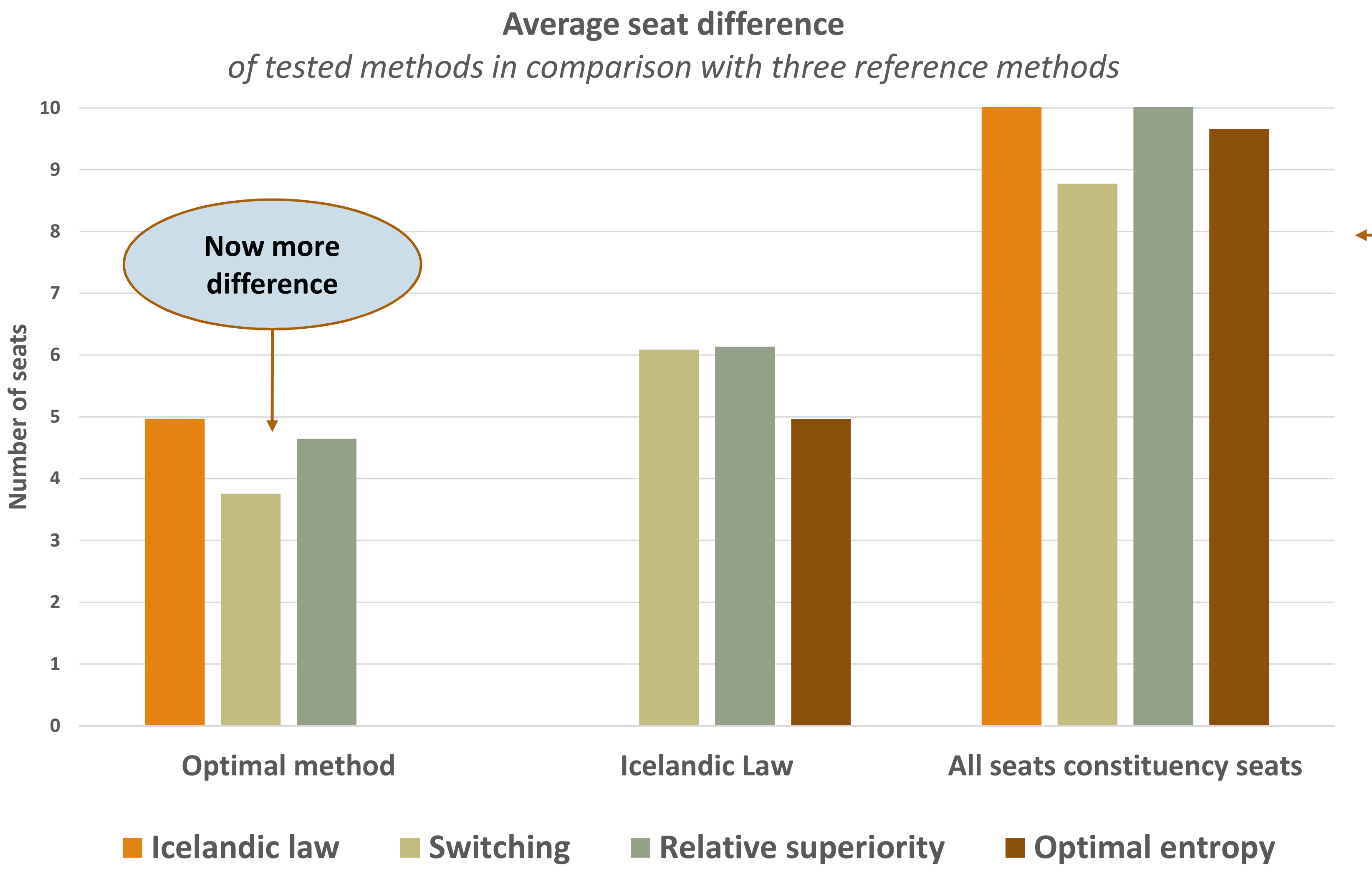
54 constituency seats

9 adjustment seats

5% threshold

All based on d'Hondt

1000 simulations



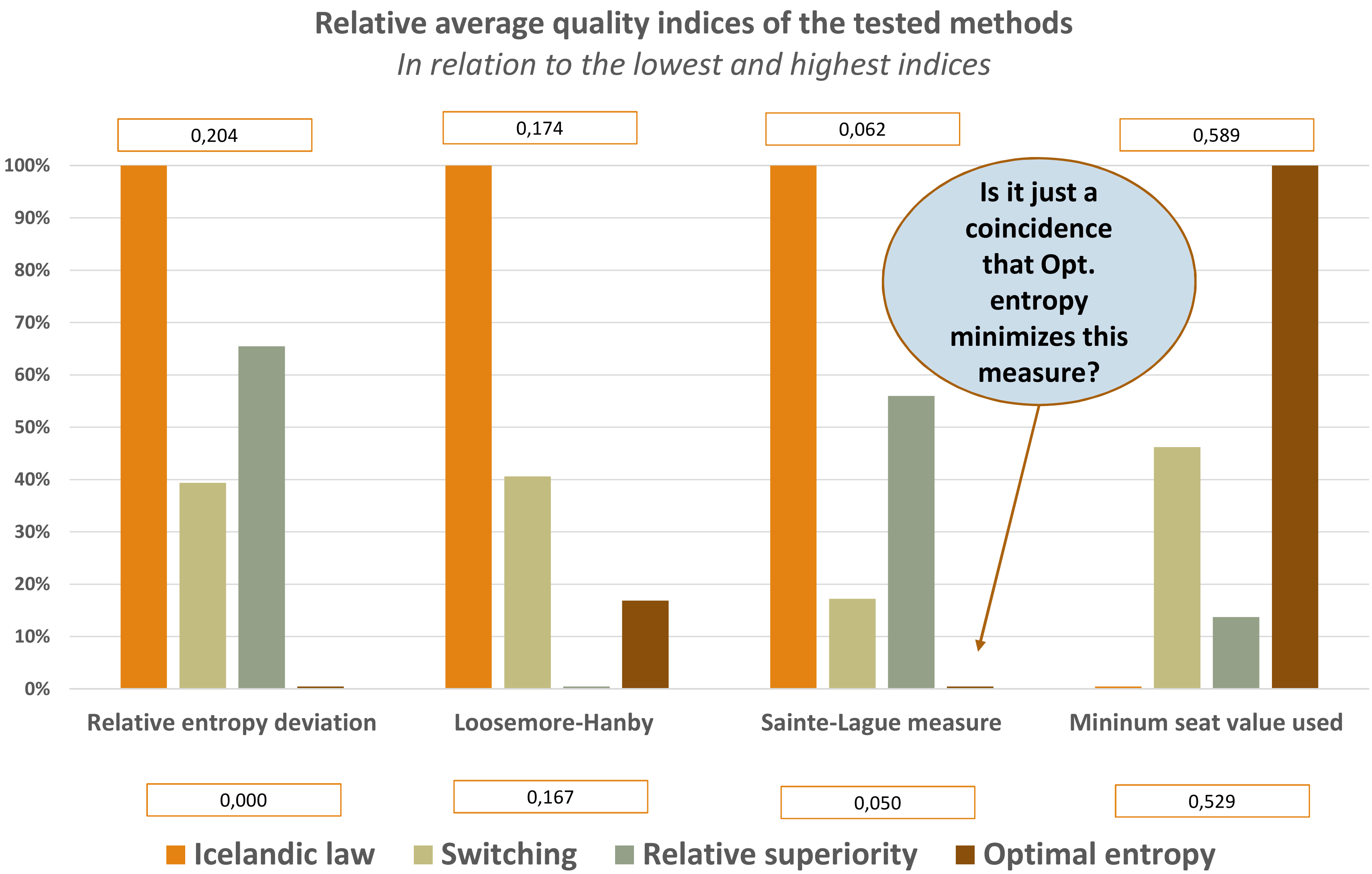
0 constituency seats

63 adjustment seats

5% threshold

All based on d'Hondt

1000 simulations



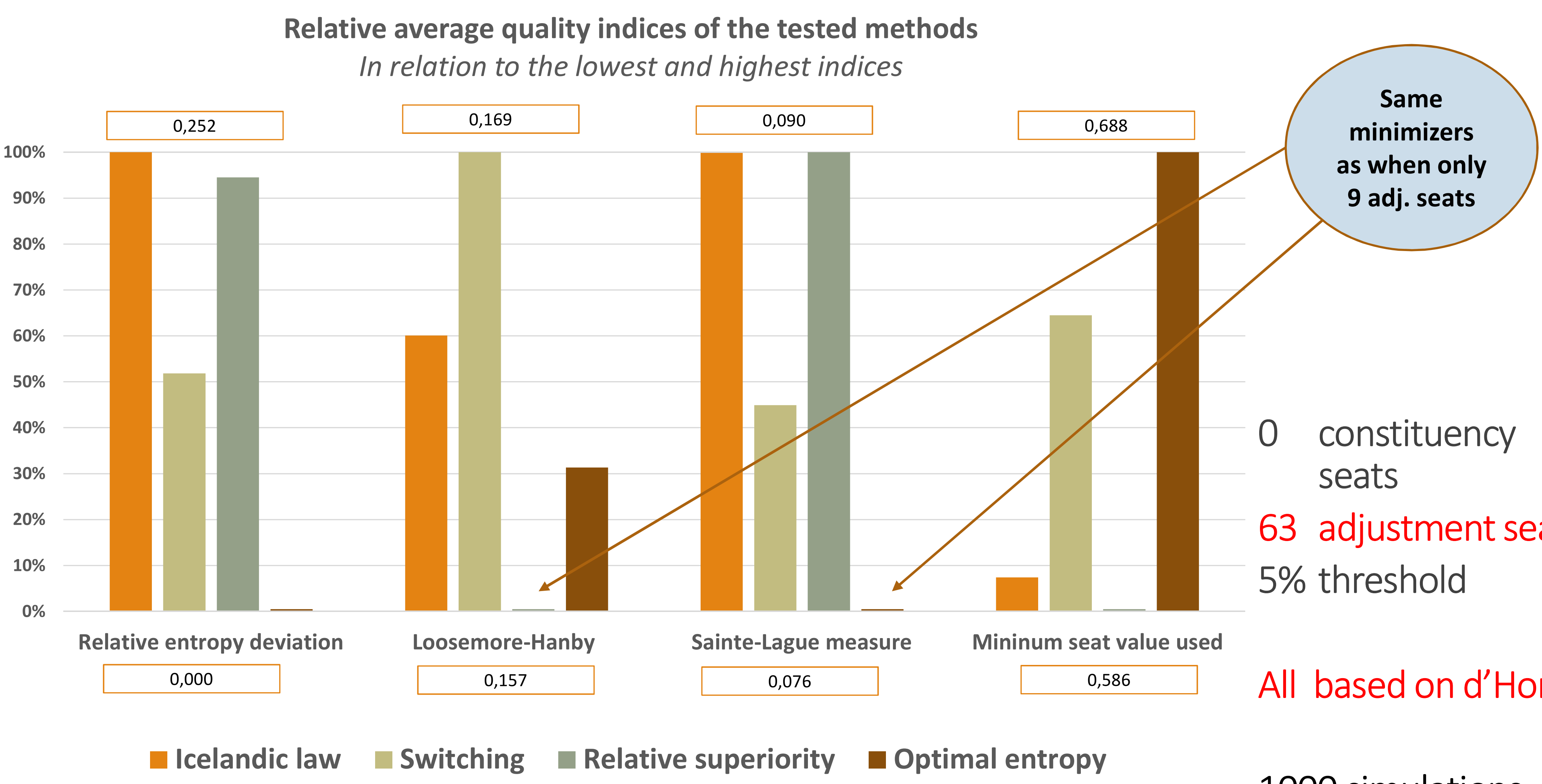
54 constituency seats

9 adjustment seats

5% threshold

All based on d'Hondt

1000 simulations



0 constituency seats

63 adjustment seats

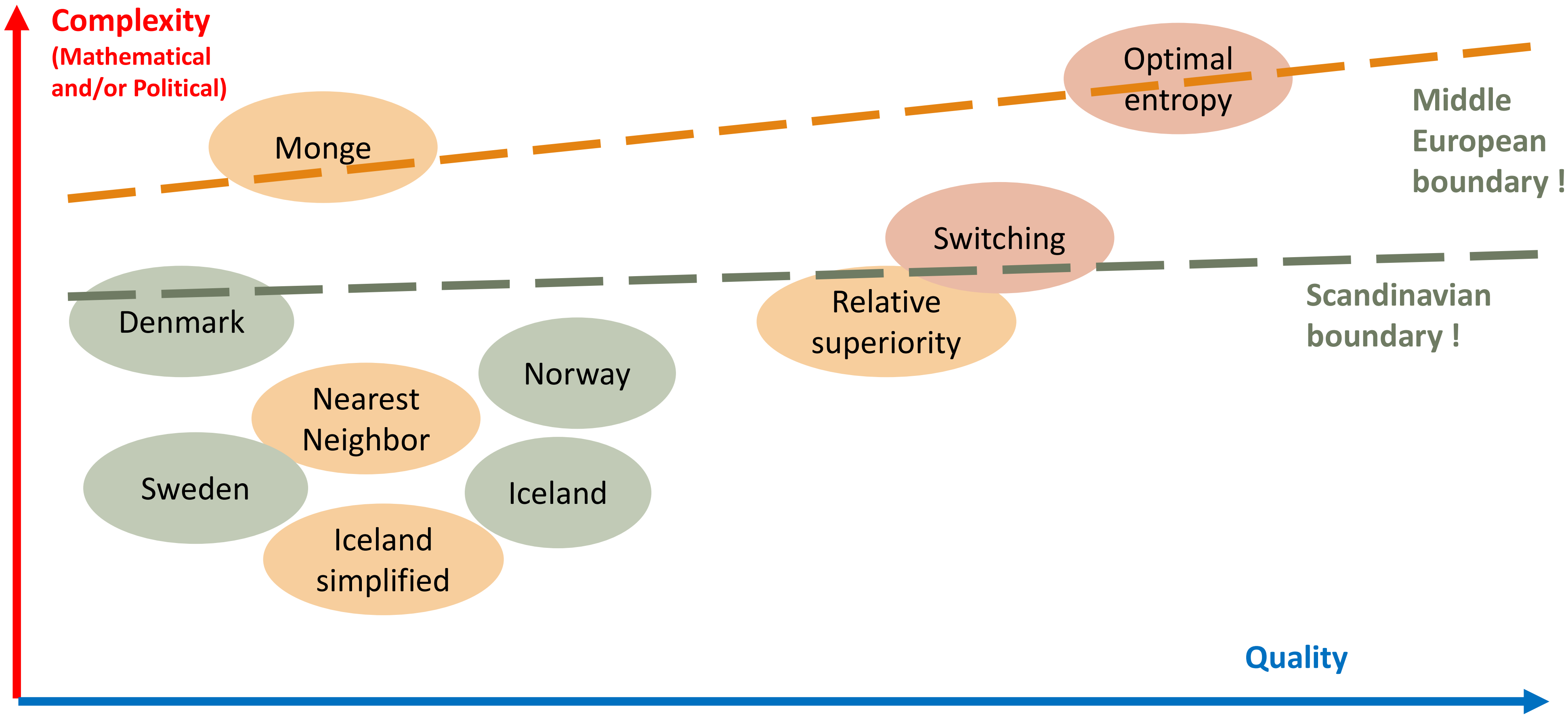
5% threshold

All based on d'Hondt

1000 simulations

Conclusions

Existing Recursive Iterative



Next steps

Partnership

is very welcome !

Contact:

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Theory

- Prove which of the B&D axioms are fulfilled by the proposed methods; and/or find counterexamples
- How to test methods effectively in particular for monotonicity and IIA, in a computationally acceptable way

Simulation program

- More user friendly
- More flexibility in electoral system design

Apportionment methods

- Develop current methods further
- Invent new – and better - methods

Ideas

- Other optimization objectives, not just entropy
- How about “negative” assignments?: Gradually excluding lists

La salud de las democracias, cualquiera que sean su tipo y su grado, depende de un mísero detalle técnico: el procedimiento electoral. Todo lo demás es secundario.

The health of democracies, of whatever type and range, depends on a wretched technical detail- electoral procedure. All the rest is secondary.

— *Jose Ortega y Gasset* —

(1883 - 1955)

(A more precise translation than used in the talk.)